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2009 J. Phys. A: Math. Theor. 42 185205

(http://iopscience.iop.org/1751-8121/42/18/185205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 185205 (21pp) doi:10.1088/1751-8113/42/18/185205

Position-dependent mass oscillators and coherent states

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Received 28 January 2009, in final form 16 March 2009
Published 9 April 2009
Online at stacks.iop.org/JPhysA/42/185205

Abstract

The solution of the Schrödinger equation for a position-dependent mass quantum system is studied in two ways. First, the interaction is found which must be applied to a mass $m(x)$ in order to supply it with a particular spectrum of energies. Second, given a specific potential $V(x)$ acting on the mass $m(x)$, the related spectrum is found. The method of solution is applied to a wide class of position-dependent mass oscillators and the corresponding coherent states are constructed. The analytical expressions of such position-dependent mass coherent states preserve the functional structure of the Glauber states.

PACS numbers: 11.30.Pb, 03.65.Ge, 03.65.Fd, 02.30.Gp

1. Introduction

The problem of calculating the energies of a quantum system endowed with position-dependent mass $m(x)$ and subjected to a given interaction represents an interface between theoretical and applied physics. Its antecedent can be identified with the concept of effective mass, introduced in the 1940s to discuss the motion of electrons or holes in semiconductors [1]. Successfully applied in describing the formation of shallow energy levels due to impurities in crystals, the effective mass theory was strongly developed in the 1950s [2]. Further insights were given in the calculation of superlattice band structures for which the band edges and the masses are position dependent. In such a context, it was stressed that the correct effective Hamiltonian consists of the kinetic term $\frac{1}{4}\left\{P^2, \frac{1}{m(x)}\right\}$ instead of the conventional expression $\frac{P^2}{2m}$ [3]. That is, the Hermiticity of the Hamiltonian is a part of the problem if the mass is not a constant.

The subject has embraced potentials other than the periodic ones over the years. Indeed, the energy bands and periodic-like interactions appearing quite naturally in semiconductor physics are substituted with point spectra and properly defined potentials in mathematical physics [4–13, 21–28]. This new perspective has inspired intense activity in the search for new exactly solvable potentials in quantum mechanics. Of particular interest, Susy-QM

(factorization or intertwining method) [14–17] and group theory [18, 19] (see also [20]) represent the most fruitful approaches to the matter [21–28]. However, the literature focuses mainly on one of the two sides of the problem. Namely, in order to get a particular spectrum, an appropriate mass function $m(x)$ and potential $V(x)$ are usually looked for. A deeper insight is necessary if one is interested in a particular mass $m(x)$ subject to a previously defined interaction rather than looking for the recovery of a specific spectrum.

In this work, we analyze the two sides of the position-dependent mass problem by following the transformation scheme of the Schrödinger equation reported in [20]. In a first step, the equation involving $m(x)$ is mapped to the equation of a constant mass m_0. After obtaining some general results, we study the eigenvalue problem connected with diverse position-dependent mass oscillators. In general, we distinguish between two fundamental kinds of oscillators. The first one is characterized by exhibiting the conventional set of equidistant energies $\hbar\omega_0(n + 1/2)$, no matter what the explicit form of $m(x)$ or $V(x)$ is. The oscillators of the second kind, on the other hand, are endowed with position-dependent mass $m(x)$ and subjected to the conventional oscillator interaction $V_{\rm osc}(x) = m_0\omega_0^2 x^2/2$. The spectra of these last oscillators depend on the explicit form of the mass function. In this way, we are able to compare the behavior of a quantum system of mass $m(x)$ with that of a particle of mass m_0 when both of them are acted on by the same oscillator-like potential. One of our motivations to analyze such oscillators is due to the fact that, as far as we know, there is a lack of results including the coherent states for position-dependent mass systems.

Originally derived for electromagnetic fields [29], the features of the standard coherent states (Glauber states) are a consequence of the oscillator dynamical algebra [30]. They are usually constructed as eigenstates of the annihilation operator but are shown to minimize the uncertainty relation between position and momentum as well. A third property is that the Glauber states are displaced versions of the ground wavefunction. For other systems, generalized coherent states (CS) can be constructed through algebraic techniques (see, e.g. [30, 31]). In general, the CS do not show all the three basic properties of the Glauber states. They have been recently studied in connection with nonlinear Susy algebras [32, 33] (see, also the reviews [34, 35]), classical motion models for the Pöschl–Teller potential [36], anharmonic vibrations in diatomic molecules [37], Landau levels [38] and the Penning trap [39]. In the present work, we introduce some families of position-dependent mass oscillator coherent states.

The paper is organized as follows. In section 2, the Schrödinger equation of a position-dependent mass system is connected with the equation of a constant mass m_0. The solutions are interrelated by a mapping for which the Hamiltonians are isospectral. The main difficulty is that the Hamiltonian of the mass m_0 includes an effective potential which, in general, makes the related equation as complicated to solve as the initial one. Here, the problem will be faced by either selecting the appropriate mass function $m(x)$ or fixing the order in which m is entangled with P in the Hamiltonian. In section 3, the previous general results are particularized to the harmonic oscillator potential. As a first result, it is shown that potentials behaving as a confining odd-root-law, $\ln^2$, or $\sinh^2$ give rise to the quantum oscillator energies. The singular oscillator $V = x^2 + \alpha x^{-2}$ is analyzed as a special case. On the other hand, it is also shown that the action of an oscillator potential on $m(x)$ involves the energies of a constant mass m_0 subject to either a confining even-power-law or the $\sinh^2$-like potentials. To deal with these last potentials, numerical approximation of the solutions will be unavoidable.

In section 4, a position-dependent mass Hamiltonian is shown to be factorized by a couple of two mutually adjoint operators, the commutator of which depends on the explicit form of $m(x)$. The initial Hamiltonian is then intertwined with a new one in such a way that they are isospectral. Factorization operators can be properly selected to work as ladders when

acting on the eigenfunctions derived in section 3. The related coherent states are constructed as eigenfunctions of the annihilation operator. These position-dependent mass CS are shown to have the same analytical form as the Glauber states. Moreover, they are also displaced versions of the ground state and minimize the uncertainty relation between P and Y, this last being the position operator in the appropriate representation. Finally, in the very last section of the paper some concluding remarks are given.

2. The eigenvalue equation

Let us consider the one-dimensional Hamiltonian

$$H_a = \tfrac{1}{2} m^a P m^{2b} P m^a + V \equiv K_a + V, \qquad 2a + 2b = -1, \tag{1}$$

where the mass $m > 0$ and the potential V are functions of the position, K_a is the kinetic term of H_a and P fulfills $[X, P] = \mathrm{i}\hbar$, with X being the position operator. We shall use $\mathcal{D}^{(a)}$ to represent the domain of definition of H_a, i.e. $\mathcal{D}^{(a)} \equiv \mathrm{Dom}(H_a)$.

In the position representation $X = x$ and $P = -\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}x}$, so we have

$$[f(x), P] = \mathrm{i}\hbar f'(x), \qquad ' \equiv \frac{\mathrm{d}}{\mathrm{d}x}. \tag{2}$$

This last commutator allows us to express the Hamiltonian (1) as follows:

$$\begin{aligned} H_a &= \frac{1}{2m} P^2 + \frac{\mathrm{i}\hbar}{2m} \left(\frac{m'}{m}\right) P - \frac{\hbar^2}{2m} \left(\frac{a}{m^2}\right) [m m'' - (2+a)(m')^2] + V \\ &= \alpha_1 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha_2 \frac{\mathrm{d}}{\mathrm{d}x} + \alpha_3, \end{aligned} \tag{3}$$

with

$$\begin{aligned} &\alpha_1 = -\frac{\hbar^2}{2m}, \qquad \alpha_2 = -\alpha_1 \left(\frac{m'}{m}\right) = \alpha_1', \\ &\alpha_3 = \alpha_1 \left(\frac{a}{m^2}\right) [m m'' - (2+a)(m')^2] + V. \end{aligned} \tag{4}$$

In order to solve the eigenvalue equation

$$H_a \psi(x) = E \psi(x), \tag{5}$$

we first rewrite the functions ψ to read

$$\psi(x) = \mathrm{e}^{g(x)} \varphi(x), \tag{6}$$

with g and φ being two functions to be determined such that

$$\int_{\mathcal{D}^{(a)}} |\psi(x)|^2 \, \mathrm{d}x = \int_{\mathcal{D}^{(a)}} |\mathrm{e}^{g(x)} \varphi(x)|^2 \, \mathrm{d}x < +\infty. \tag{7}$$

Hence, from (5) one gets

$$\alpha_1 \varphi'' + (2\alpha_1 g' + \alpha_2)\varphi' + \{\alpha_1 [g'' + (g')^2] + \alpha_2 g' + \alpha_3 - E\}\varphi = 0. \tag{8}$$

Now, we introduce a change of the independent variable x, ruled by a bijection s as follows:

$$x \mapsto y = s(x), \qquad y \mapsto x = s^{-1}(y). \tag{9}$$

The Jacobian of the transformation is given by $J = s'(x)$. If $J \neq 0$ at a point x, the *inverse function theorem* indicates that the map s is 1–1 and onto in some neighborhood of x (see, e.g. [40 pp 91]). In this way, to construct a well-defined bijection s we first ask for the involved Jacobian s' to be free of zeros. On the other hand, let f be a function of x. Then we have

$$f(x) = f(s^{-1}(y)) = [f \circ s^{-1}](y) \equiv f_*(y). \tag{10}$$

Thus, f_* is the representation of the function f in the y-space. In a similar manner we find that f is the representation of f_* in the x-space: $f = [f_* \circ s]$. Hereafter, and whenever there will be no confusion, we drop the subindex '*' from the functions in the y-representation. The straightforward calculation departing from equation (8) leads to

$$\alpha_1(y')^2\ddot{\varphi} + [\alpha_1 y'' + (2\alpha_1 g' + \alpha_2)y']\dot{\varphi} + \{\alpha_1[g'' + (g')^2] + \alpha_2 g' + \alpha_3 - E\}\varphi = 0, \tag{11}$$

where $\dot{f} \equiv \mathrm{d}f/\mathrm{d}y$. This last equation acquires a simple form if the coefficients of $\ddot{\varphi}$ and $\dot{\varphi}$ are respectively a constant c_0^2 (expressed in appropriate units) and zero. Thereby one has a system of equations

$$\alpha_1(y')^2 = c_0^2, \qquad \alpha_1 y'' + (2\alpha_1 g' + \alpha_2)y' = 0, \tag{12}$$

the solution of which defines the forms of g and y in terms of the mass position dependence:

$$g(x) = \ln\left[\frac{m(x)}{m_0}\right]^{1/4}, \qquad y = \int \mathrm{e}^{2g(x)}\,\mathrm{d}x + y_0. \tag{13}$$

Here, m_0 and y_0 are integration constants (we shall take, for simplicity, $y_0 = 0$) and the constant c_0, introduced in (12), has been written as $c_0 = \mathrm{i}\hbar/\sqrt{2m_0}$. These last results in (11) reduce the initial eigenvalue equation (5) to the following one:

$$H_{\mathrm{eff}}^{(a)}\varphi(y) := \left[-\left(\frac{\hbar^2}{2m_0}\right)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V_{\mathrm{eff}}^{(a)}(y)\right]\varphi(y) = E\varphi(y), \tag{14}$$

where the function

$$V_{\mathrm{eff}}^{(a)} := V - \left(\frac{\hbar^2}{2m^3}\right)\left[\left(\frac{1}{4} + a\right)mm'' - \left\{\frac{7}{16} + a(2+a)\right\}(m')^2\right] \tag{15}$$

plays the role of an 'effective potential' and depends on the explicit expressions for the mass m and the initial potential V, both of them in the y-representation. In particular, if the mass function m is a constant then we have $V_{\mathrm{eff}}^{(a)} = V$. In general, $m(x)$ could lead to a very complex function $V_{\mathrm{eff}}^{(a)}(y)$ for which the new equation (14) is as complicated to solve as the initial one (5). Hence, at this stage, the main simplification is the avoiding of undesirable mass factors in the derivative term so that techniques to solve the conventional Schrödinger equation can be applied.

Given a solution φ_* of (14), according to (6), (9) and (13), the function ψ is

$$\psi(x) = J^{1/2}[\varphi_* \circ s](x), \qquad J = \left[\frac{m(x)}{m_0}\right]^{1/2}. \tag{16}$$

Therefore, we have

$$\int_{\mathcal{D}^{(a)}} |\psi(x)|^2\,\mathrm{d}x = \int_{\mathcal{D}^{(a)}} J|\varphi_*(s(x))|^2\,\mathrm{d}x \quad \longleftrightarrow \quad \int_{\mathcal{D}_{\mathrm{eff}}^{(a)}} |\varphi_*(y)|^2\,\mathrm{d}y, \tag{17}$$

with $\mathcal{D}_{\mathrm{eff}}^{(a)} \equiv \mathrm{Dom}\big(H_{\mathrm{eff}}^{(a)}\big)$. That is, by getting the square-integrable eigenfunctions of $H_{\mathrm{eff}}^{(a)}$ one is able to obtain the square-integrable eigenfunctions of H_a and vice versa. Moreover, from (5) and (14) we note that φ_* and ψ share the same eigenvalue E. Thus, H_a and $H_{\mathrm{eff}}^{(a)}$ are isospectral operators; we write $\mathrm{Sp}(H_a) = \mathrm{Sp}\big(H_{\mathrm{eff}}^{(a)}\big)$.

Note that equations (15) and (16) are consistent with the results reported in [5, 23, 25]. With regard to our approach, there are still a couple of special cases leading to further simplifications. Namely, one can get $V_{\mathrm{eff}}^{(a)}(y) = V(y)$ by selecting the appropriate function $m(x)$ or by properly fixing the value of a, as we are going to show.

2.1. Mass-dependent null terms (MDNT)

Let us look for a mass function m such that $V_{\text{eff}}^{(a)} - V = 0$ in equation (15). Thus, we should solve the nonlinear, second-order differential equation:

$$c_1 m m'' + c_2 (m')^2 = 0, \qquad 2c_1 = \tfrac{1}{16} - a^2 - c_2 = \tfrac{1}{2} + 2a. \tag{18}$$

A brief examination yields

$$m(x;a) = m_0 (x_0 + \lambda x)^{-4/(3+4a)}, \qquad a \neq -3/4 \tag{19}$$

as the simplest solution with x_0 and λ being constants to be fixed. We have to distinguish between two general cases, as follows:

(I) If $a < -3/4$, then $m(x;a)$ has a zero at $x = t_0 \equiv -x_0/\lambda$.
(II) If $a > -3/4$, then $m(x;a)$ is singular at $x = t_0$.

The first case will be omitted to avoid ill-defined operators H_a and inappropriate mappings s as well. Indeed, if $a < -3/4$, the kinetic term K_a in equation (1) diverges and the Jacobian J in (16) is zero at $x = t_0$. On the other hand, for $a > -3/4$ the integrability of $\sqrt{m(x;a)}$ in equation (13) depends on the value of a. In particular, if $a = a_0 \equiv -1/4$ then the mapping $x \mapsto y_{(0)}$ is ruled by the function

$$y_{(0)} = s_{(0)}(x) = \frac{\ln(x_0 + \lambda x)}{\lambda}, \qquad x \geqslant t_0, \tag{20}$$

with

$$m_{(0)}(x) \equiv m(x;a_0) = \frac{m_0}{(x_0 + \lambda x)^2}. \tag{21}$$

This last expression of $m(x)$ is connected with the revival wave packets in a position-dependent mass infinite well [10]. Here, the Jacobian reads as $J_{(0)} = 1/(x_0 + \lambda x)$, so that the bijection $s_{(0)}$ is well defined for all $x \geqslant t_0$ and arbitrary real values of x_0 and λ. Then, in general $\mathcal{D}^{(a_0)} \subseteq [t_0, +\infty)$ and $\mathcal{D}_{\text{eff}}^{(a_0)} \subseteq \mathbb{R}$. The explicit form of the domains of definition $\mathcal{D}^{(a_0)}$ and $\mathcal{D}_{\text{eff}}^{(a_0)}$ depends on V while the inverse function reads as

$$x = s_{(0)}^{-1}(y) = \frac{\mathrm{e}^{\lambda y_{(0)}} - x_0}{\lambda}. \tag{22}$$

On the other hand, if $a \neq a_0$ the new variable is given by

$$y = s(x;a) = \frac{(x_0 + \lambda x)^\eta}{\lambda \eta}, \qquad \eta = \left(\frac{1+4a}{3+4a}\right). \tag{23}$$

The appropriate mapping $s(x;a)$ is fixed by looking for the values of a such that either $\eta = 2n+1$ or $\eta^{-1} = 2n+1$, $n = 1, 2, \ldots$. In the former case we arrive at the discrete set integrated by the points $-\frac{3n+1}{4n} = -1, -\frac{7}{8}, -\frac{5}{6}, \ldots < -\frac{3}{4}$. However, each one of these possible values of a belongs to case (I) discussed above and must be omitted. Now, if $\eta^{-1} = 2n+1$ we obtain the points $\frac{1-n}{4n} = 0, -\frac{1}{8}, -\frac{1}{6}, \ldots > -\frac{1}{4}$, which belong to case (II) we are interested in. Henceforth, the mapping $x \mapsto y_{(n)}$ is established from equation (23) as follows:

$$y_{(n)} = s(x;a_n) \equiv s_{(n)}(x) = \left(\frac{2n+1}{\lambda}\right)(x_0 + \lambda x)^{1/(2n+1)}, \qquad a_n \neq a_0, \tag{24}$$

while the corresponding inverse transformation is ruled by

$$x = s_{(n)}^{-1}(y_{(n)}) = \frac{1}{\lambda}\left[\left(\frac{\lambda y_{(n)}}{2n+1}\right)^{2n+1} - x_0\right], \qquad a_n \neq a_0. \tag{25}$$

Table 1. Special mass functions $m(x)$ and orderings of the kinetic term $K_a(x)$ leading to the Hamiltonians H_a and $H_{\rm eff}^{(a)}$, with $V_{\rm eff}^{(a)}(y) = V(y)$ and $\mathrm{Sp}(H_a) = \mathrm{Sp}(H_{\rm eff}^{(a)})$. In all cases, the definite domain is fixed by V.

$a_0 = -\frac{1}{4}$	$H_{a_0} = \frac{1}{2}m_{(0)}^{-1/4}Pm_{(0)}^{-1/2}Pm_{(0)}^{-1/4} + V(x)$	$H^{(a_0)} = -\left(\frac{\hbar^2}{2m_0}\right)\frac{\mathrm{d}^2}{\mathrm{d}y_{(0)}^2} + V(y_{(0)})$
$m_{(0)}(x) = \frac{m_0}{(x_0+\lambda x)^2}$		
$\lambda y_{(0)} = \ln(x_0 + \lambda x)$	$\mathcal{D}^{(a_0)} \subseteq [t_0, +\infty)$	$\mathcal{D}_{\rm eff}^{(a_0)} \subseteq \mathbb{R}$
$a_1 = 0$	$H_{a_1} = \frac{1}{2}Pm_{(1)}^{-1}P + V(x)$	$H^{(a_1)} = -\left(\frac{\hbar^2}{2m_0}\right)\frac{\mathrm{d}^2}{\mathrm{d}y_{(1)}^2} + V(y_{(1)})$
$m_{(1)}(x) = \frac{m_0}{(x_0+\lambda x)^{4/3}}$		
$\lambda y_{(1)} = 3(x_0 + \lambda x)^{1/3}$	$\mathcal{D}^{(a_1)} \subseteq \mathbb{R}$	$\mathcal{D}_{\rm eff}^{(a_1)} \subseteq \mathbb{R}$
$a_n = \frac{(1-n)}{4n}, n \in \mathbb{N}$	$H_{a_n} = \frac{1}{2}m_{(n)}^{a_n}Pm_{(n)}^{-1-2a_n}Pm_{(n)}^{a_n} + V(x)$	$H^{(a_n)} = -\left(\frac{\hbar^2}{2m_0}\right)\frac{\mathrm{d}^2}{\mathrm{d}y_{(n)}^2} + V(y_{(n)})$
$m_{(n)}(x)$ (see, equation (26))		
$y_{(n)}$ (see, equation (24))	$\mathcal{D}^{(a_n)} \subseteq \mathbb{R}$	$\mathcal{D}_{\rm eff}^{(a_n)} \subseteq \mathbb{R}$
$a = -\frac{1}{4}$	$H_{-1/4} = \frac{1}{2}m^{-1/4}Pm^{-1/2}Pm^{-1/4} + V(x)$	$H = -\left(\frac{\hbar^2}{2m_0}\right)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V(y)$
$m(x) > 0$		
$y = \int (m/m_0)^{1/2}\mathrm{d}x$	$\mathcal{D}^{(-1/4)} \subseteq \mathbb{R}$	$\mathcal{D}_{\rm eff}^{(-1/4)} \subseteq \mathbb{R}$

The expression for the mass function (19) in terms of $a_n \neq a_0$ reduces to

$$m_{(n)}(x) \equiv m(x; a_n) = \frac{m_0}{(x_0 + \lambda x)^{4n/(2n+1)}}, \qquad n \in \mathbb{N}. \tag{26}$$

Note that $J_{(n)} = (x_0 + \lambda x)^{-2n/(2n+1)}$. Hence $J_{(n)} \neq 0$ for all $x \in \mathbb{R}$ and arbitrary real values of x_0 and λ. As a consequence $\mathcal{D}^{(a_n)} \subseteq \mathbb{R}$ and $\mathcal{D}_{\rm eff}^{(a_n)} \subseteq \mathbb{R}$. To embrace $a_{n>0}$ and a_0 into the same notation, let us introduce the set

$$\mathcal{A} = \left\{a_0 = -1/4, a_n = \frac{1-n}{4n}\right\}, \qquad n \in \mathbb{N}. \tag{27}$$

Then, if $a \in \mathcal{A}$ the position-dependent mass operator H_{a_n} is mapped to a conventional Hamiltonian $H^{(a_n)}$ in the $y_{(n)}$-representation and vice versa (see table 1):

$$H_{a_n} \leftrightarrow H_{\rm eff}^{(a_n)} \equiv H^{(a_n)} = -\left(\frac{\hbar^2}{2m_0}\right)\frac{\mathrm{d}^2}{\mathrm{d}y_{(n)}^2} + V(y_{(n)}), \tag{28}$$

with $\mathrm{Sp}(H_{a_n}) = \mathrm{Sp}(H^{(a_n)})$. We shall take full advantage of this last property in the following sections.

2.2. Mass-independent null terms (MINT)

A simple inspection of equation (15) shows that $V_{\rm eff}^{(-1/4)}(y) = V(y)$, no matter what the explicit form of the mass function $m(x)$ is—assuming the latter is well defined. That is, by fixing $a = -1/4$ we get

$$H_{-1/4} \leftrightarrow H_{\rm eff}^{(-1/4)} \equiv H = -\left(\frac{\hbar^2}{2m_0}\right)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + V(y). \tag{29}$$

In particular, if $m = m_{(0)}$ then H in (29) is the same as $H^{(a_0)}$ with $\mathcal{D}^{(-1/4)} \equiv \mathrm{Dom}(H_{-1/4}) = \mathcal{D}^{(a_0)}$ and $\mathrm{Dom}(H) = \mathcal{D}_{\rm eff}^{(a_0)}$. A similar situation occurs if $m = m_{(n)}$ (see table 1).

Besides the mass functions derived in the previous section, a regular expression for m has been recently introduced in [27] (see also [7]). This is given by the function m_R:

$$m_R(x) = \frac{m_0}{1 + (\lambda x)^2}, \qquad \lambda \in \mathbb{R}, \tag{30}$$

with

$$s_R(x) = \frac{\operatorname{arcsinh}(\lambda x)}{\lambda}, \qquad s_R^{-1}(y) = \frac{\sinh(\lambda y)}{\lambda}. \tag{31}$$

The corresponding Jacobian $J = 1/\sqrt{1 + (\lambda x)^2}$ is nonzero for all $x \in \mathbb{R}$ and arbitrary values of λ. Hence $\mathcal{D}^{(-1/4)} \subseteq \mathbb{R}$ and $\mathcal{D}_{\text{eff}}^{(-1/4)} \subseteq \mathbb{R}$. The main aspects of these last results are summarized in table 1. As a final remark, the mass (30) appeared in the construction of the Wigner functions connected with a class of position-dependent oscillators [13]. Other interesting mass functions are

$$m_w(x) = \left(\frac{w + x^2}{1 + x^2}\right)^2, \qquad m_c(x) = cx^2. \tag{32}$$

They have already been studied in [22] and recently in e.g. [6]. Note that $J_c(x = 0) = 0$ and $J_w \neq 0 \ \forall x \in \mathbb{R}$ and $w > 0$. In the following sections we shall study specific forms of the potentials $V(x)$ and $V_*(y)$, which represent oscillator-like interactions for a position-dependent mass quantum system.

3. Two kinds of position-dependent mass oscillators

We are going to work with the eigenvalue equation (14) such that $V_{\text{eff}}^{(a)} = V$ by either the MDNT or the MINT cases described in the previous sections. Although our approach holds for any well-defined potential V, we shall focus on the linear harmonic oscillator in two general situations.

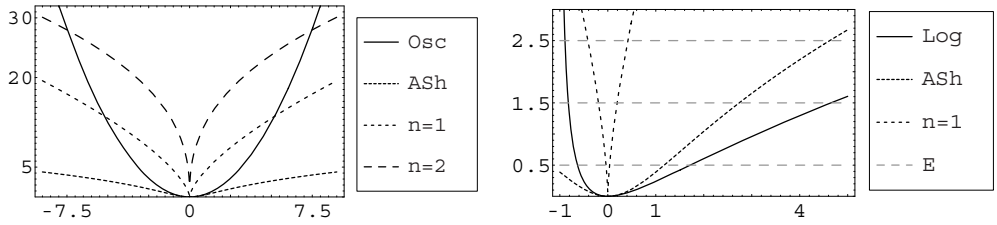
(i) Departing from a given interaction $V(x)$ and a mass function $m(x)$, we arrive at the conventional linear harmonic oscillator problem in the y-representation. That is, the new potential reads as $V_*(y) = \frac{m_0\omega_0^2}{2}y^2$, with ω_0 being the natural frequency of oscillation. Since $V(x)$ and $V_*(y)$ are isospectral they share the eigenvalues defined by $E_n = \hbar\omega_0(n + 1/2), n = 0, 1, 2, \ldots$. We shall refer to these potentials as *oscillators of the first kind*.

(ii) Departing from the linear harmonic oscillator interaction $V(x) = \frac{m_0\omega_0^2}{2}x^2$ and a mass function $m(x)$ we arrive at the eigenvalue equation connected with the new potential $V_*(y)$. Since $V(x)$ and $V_*(y)$ are isospectral, we solve the (conventional) Schrödinger equation in the y-representation to construct the solutions of the initial oscillator-like, position-dependent mass problem. We shall refer to these potentials as *oscillators of the second kind*.

3.1. Oscillators of the first kind

Let us take $V_*(y) = \frac{m_0\omega_0^2}{2}y^2$ as the y-representation of the initial potential $V(x)$. Then $\mathcal{D}_{\text{eff}}^{(a)} = \mathbb{R}$ and all the mappings MDNT and MINT can be applied (see table 1). It is convenient to introduce a dimensionless notation as follows:

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{dy}^2} + \frac{\mathrm{y}^2}{2} - \mathrm{E}\right]\varphi(\mathrm{y}) = 0, \qquad \mathrm{y} = y\left(\frac{\hbar}{m_0\omega_0}\right)^{-1/2} \equiv y\alpha, \qquad \mathrm{E} = \frac{E}{\hbar\omega_0}. \tag{33}$$



(*a*) (*b*)

Figure 1. (*a*) The odd root-law potential (35) with $n = 1$, $n = 2$ and $x_0 = 0$ besides the regular one (ASh) defined in equation (38). The harmonic oscillator potential (Osc) is depicted as a reference. (*b*) The square-logarithmic potential (Log) defined in equation (35) with $x_0 = 1$ and $\mathcal{D}^{(a_0)} = [-1, +\infty)$. Potentials (Ash) and ($n = 1$) as well as the first three energy levels (E) are also depicted. The vertical and horizontal axes are respectively in $\hbar\omega_0$ and dimensionless units.

Then, the solutions read as

$$\varphi_n(\mathrm{y}) = \frac{H_n(\mathrm{y})\,\mathrm{e}^{-\mathrm{y}^2/2}}{\sqrt{2^n \pi^{1/2} n!}}, \qquad H_n(\mathrm{y}) = (-1)^n\,\mathrm{e}^{\mathrm{y}^2/2}\frac{\mathrm{d}^n}{\mathrm{dy}^n}\,\mathrm{e}^{-\mathrm{y}^2/2}, \qquad \mathrm{E}_n = n + \frac{1}{2}. \tag{34}$$

Next, we are going to solve the initial position-dependent mass problem in terms of these results.

3.1.1. MDNT case. Let $m(x; a)$ be the mass function with $a \in \mathcal{A}$, that is $m = m_{(n)}$, $n = 0, 1, 2, \ldots$. From equations (20) and (24) we know that the initial potential reads as

$$V_{(n)}(x) = [V_* \circ s_{(n)}](x) = \frac{m_0\omega_0^2}{2\lambda^2}\begin{cases}(2n+1)^2(x_0+\lambda x)^{\frac{2}{(2n+1)}}, & n \in \mathbb{N}, \quad x \in \mathbb{R} \\ \ln^2(x_0+\lambda x), & n = 0, \quad x \in [t_0, +\infty).\end{cases} \tag{35}$$

A dimensional analysis shows that $\lambda = \lambda_0\alpha$, with λ_0 being a constant ($\lambda_0 = 1$ for simplicity). The behavior of potential (35) for $n = 1$ and $n = 2$ is contrasted with the well-known curve of the harmonic oscillator potential in figure 1(*a*); the case $n = 0$ is depicted in figure 1(*b*). In both cases, as we have previously noted, the involved spectrum is given by $E_k = \hbar\omega_0(k + 1/2)$, $k = 0, 1, 2, \ldots$, while their eigenfunctions respectively read as

$$\psi_k(x) = \frac{H_k\left[(2n+1)(x_0+\alpha x)^{\frac{1}{(2n+1)}}\right]}{(x_0+\alpha x)^{\frac{n}{(2n+1)}}\sqrt{2^k\pi^{1/2}k!}}\exp\left[-\frac{(2n+1)^2}{2}(x_0+\alpha x)^{\frac{2}{(2n+1)}}\right], \qquad n \in \mathbb{N}, \tag{36}$$

and

$$\psi_k(x) = \frac{H_k[\ln(x_0+\alpha x)]}{\sqrt{(x_0+\alpha x)2^k\pi^{1/2}k!}}\,\mathrm{e}^{-\frac{1}{2}\ln^2(x_0+\alpha x)}, \qquad n = 0, \tag{37}$$

with x running in the domains indicated in (35). Thus, the energy spectrum of a position-dependent mass quantum system which is subject to either the action of a confining odd-root-law potential $V_{(n)}(x) \propto (\alpha x)^{2/(2n+1)}$, $n \in \mathbb{N}$, or to a square-logarithmic interaction $V_{(0)}(x) \propto \ln^2(\alpha x)$, is ruled by the quantization of the conventional harmonic oscillator energy if the mass function is respectively taken as $m_{(n)}$ or $m_{(0)}$. It is worthwhile to mention that the wave functions (36) and (37) inherit the singular point $x = t_0$ of the corresponding masses. Since the kinetic and potential energies go to zero as $x \to t_0$, it is natural that the probability amplitude becomes important in the vicinity of t_0. Yet, the wave functions are square integrable and can be normalized consistently with equations (7) and (17).

3.1.2. MINT case. Let $m_R(x)$ be the mass function with $\lambda = \alpha$. Then, the spectrum of the potential

$$V(x) = [V_* \circ s](x) = \frac{\hbar\omega_0}{2}\operatorname{arcsinh}^2(\alpha x) \tag{38}$$

is given by $E_k = \hbar\omega_0(k + 1/2)$, $k = 0, 1, 2, \ldots$, and the involved eigenfunctions read as

$$\psi_k(x) = \left[\frac{m(x)}{2^{2k}\pi(k!)^2 m_0}\right]^{1/4} H_k[\operatorname{arcsinh}(\alpha x)]\, \mathrm{e}^{-\frac{1}{2}\operatorname{arcsinh}^2(\alpha x)}, \qquad k = 0, 1, 2, \ldots. \tag{39}$$

In figure 1 the global behavior of potential (38) is shown in contrast with the curve of the harmonic oscillator one.

As we can see, one is able to identify the kind of interaction $V(x)$ which has to be applied to a quantum system of position-dependent mass $m(x)$ in order supply it with a specific, well-known, spectrum $\mathrm{Sp}(H^{(a)})$. For instance, if $\mathrm{Sp}(H^{(a)}) = \{\hbar\omega_0(n + 1/2)\}_{n=0}^{+\infty}$, we have shown that the system has to be subject to potentials behaving as a confining odd-root-law, $\ln^2$ or $\sinh^2$, whenever the mass function is respectively defined by (21), (26) or (30). A deeper insight is necessary if one is interested in a position-dependent mass $m(x)$, subject to a particular interaction $V(x)$, rather than in the recovery of a given spectrum. That is, what sort of spectrum is expected by applying an oscillator-like interaction to a quantum system of mass function $m(x)$? We will face this problem in the following section.

3.2. Oscillators of the second kind

In this section, we analyze the effects on the energy spectrum produced by a position dependence of the mass. In other words, how different is the spectrum of a system of mass $m(x)$ from that of a particle of mass m_0 when both of them are under the action of the same potential $V(x)$? As before, we shall focus on the simplest case of the linear harmonic oscillator interaction.

Let $V_{\mathrm{osc}}(x) = \frac{m_0\omega_0^2}{2}x^2$ be the initial potential. Note that $\mathrm{Dom}(V_{\mathrm{osc}}) = \mathbb{R}$ requires $\mathcal{D}^{(a)} = \mathbb{R}$. However $\mathcal{D}^{(a_0)} \subseteq [t_0, +\infty)$, so that a must be different from a_0 (see table 1). The case $a = a_0$ will be analyzed in section 3.2.3.

3.2.1. MDNT case. Let $m_{(n)}(x)$ be the mass function with $n \in \mathbb{N}$ and $\lambda = \alpha$. The initial potential $V_{\mathrm{osc}}(x)$ behaves as an even-power-law function in the $y_{(n)}$-space (see figure 2):

$$V_*(y; n) = \left[V_{\mathrm{osc}} \circ s_{(n)}^{-1}\right](y) = \frac{\hbar\omega_0}{2}\left[\left(\frac{\alpha y}{2n+1}\right)^{2n+1} - x_0\right]^2, \qquad n \in \mathbb{N}, \tag{40}$$

where the label '(n)' has been dropped from the y-coordinate for simplicity. Hereafter we shall take $x_0 = 0$. Note that $V_*(y; n) \to 0$ as $n \to +\infty$ and $V_*(y; n) \to \hbar\omega_0(\alpha y)^2/2$ as $n \to 0$. Thus, the family of potentials (40) is delimited by the free particle and the harmonic oscillator potentials (remember that $n = 0$ and $n \to +\infty$ are forbidden in equation (24)). Such behavior is shown in figure 2.

Let us emphasize that although this kind of potential is not analytically solvable, they have deserved special attention in pedagogical as well as in research papers over the years. For instance, their WKB energy levels have been shown to depend on the power of the potential [41] and the involved scale invariance has been studied in terms of the Lie method [42]. The polarizability of a particle in a power-law potential due to the presence of a constant force and the wave packet revivals in such potentials, on the other hand, have been exhaustively studied in [43, 44] respectively.

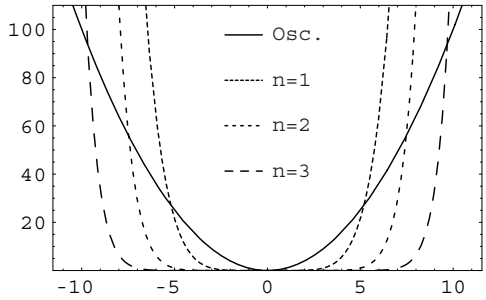


Figure 2. Three members of the family of power-law potentials (40). The conventional harmonic oscillator potential (Osc) is recovered for the forbidden value $n = 0$ and the family goes to the free particle case for $n \to +\infty$.

In general, the roots of $V_*(y; n) = V_{\rm osc}(y)$ define a region $(-y_c, y_c) \subset \mathcal{D}_{\rm eff}^{(a_n)}$ in which the potential V_* grows up more slowly than $V_{\rm osc}$. The geometry of these last curves in $(-y_c, y_c)$ suggests that the spectrum of V_* will be integrated by energy levels which are below the corresponding oscillator energies. The behavior of the curves in the complementary region $\mathcal{D}_{\rm eff}^{(a_n)} \setminus (-y_c, y_c)$ is such that the energy levels are expected to be above the oscillator ones. To verify our statement, let us calculate the eigenvalues of $V_*(y; n)$ by means of the energy quantization condition of the WKB method:

$$\int_{-y_0}^{+y_0} \sqrt{2m_0[E(n) - V(y; n)]}\,{\rm d}y = \pi\hbar(k + 1/2), \qquad k = 0, 1, 2, \ldots, \tag{41}$$

with $\pm y_0 = \pm\left(\frac{2n+1}{\alpha}\right)(2{\rm E}(n))^{1/(4n+2)}$ being the classical (symmetric) turning points and $E(n)$ the energy connected with the potential $V_*(y; n)$ for a given $n \in \mathbb{N}$. The change of variable $y = y_0 z$ reduces the integral equation (41) to (compare with [41, 42]):

$$E_k(n) = \frac{\hbar\omega_0}{2}\left[\frac{\pi}{j_n}\frac{(k + 1/2)}{(2n + 1)}\right]^{\frac{2n+1}{n+1}}, \tag{42}$$

where the constant

$$j_n = \int_{-1}^{1} \sqrt{1 - z^{4n+2}}\,{\rm d}z = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{4n+2}\right)}{2(n + 1)\Gamma\left(\frac{n+1}{2n+1}\right)} \tag{43}$$

is such that $j_n \to 2$ as $n \to +\infty$ and $j_0 = \pi/2$. Figure 3 shows the spectrum curves of three members of the family (40) compared with the energy spectrum curve of the harmonic oscillator. Note that the energy levels become closer to each other as the label n increases (free particle case). That is, if $n \gg 1$ then $E_k(n) \propto [(k + 1/2)/n]^2$. On the other hand, for the forbidden value $n = 0$ we have the oscillator spectrum $E_k(0) = \hbar\omega_0(k + 1/2)$, as was expected. The corresponding set of eigenfunctions, in turn, can be numerically constructed or analyzed by using improved versions of the WKB method like that discussed in [45].

Now, let us take one of the allowed values of n. The root of equation $E_k(n) = E_k(0)$ is given by

$$k_c(n) = \frac{1}{2}\left\{\left[\frac{(2n + 1)\Gamma\left(\frac{1}{4n+2}\right)}{\sqrt{\pi}(n + 1)\Gamma\left(\frac{n+1}{2n+1}\right)}\right]^{\frac{2n+1}{n}} - 1\right\}. \tag{44}$$

The ceiling function $\lceil k_c \rceil$ of $k_c(n)$ defines a subset of ${\rm Sp}(V_*(y; n)) = \{E_k(n)\}_{k=0}^{+\infty}$ for which $E_k(n) < E_k(0)\ \forall\ k < \lceil k_c \rceil$. The larger the value of n, the bigger the set of eigenvalues $E_k(n)$

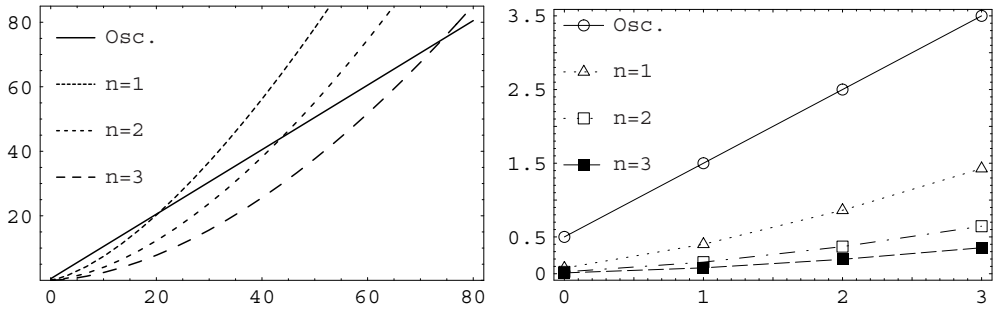


Figure 3. Left: the spectrum curves $E_k(n)$ of the power-law potential (40) for $n = 1, 2, 3$ and the spectrum curve of the harmonic oscillator (Osc); all of them are depicted in $\hbar\omega_0$ units as a function of k. Note the points in which $E_k(n) = E_k(0)$. Right: details of the first four energy levels.

delimited by $E_k(0)$. The complementary set is then such that $E_k(n) > E_k(0)\ \forall\ k \geqslant \lceil k_c \rceil$; details are shown in figure 3.

In conclusion, the oscillator of the second kind defined by the pair $(V_{\rm osc}, m_{(n)}), n \in \mathbb{N}$, shares its spectrum with a particle of mass m_0 subject to an even-power-law potential of the form $V_*(y_{(n)}; n) \propto [y_{(n)}/(2n+1)]^{4n+2}$. When contrasted with a conventional oscillator of mass m_0, the energy spectrum of the pair $(V_{\rm osc}, m_{(n)})$ is a *distorted* version of the oscillator one. The shape and amount of the distortion are respectively dictated by equation (42) and $k_c(n)$, as the latter was defined in (44). That is, the distortion is stronger for larger values of $|\lceil k_c \rceil - k|$ in equation (42).

3.2.2. MINT case. Let $m_R(x)$ be the mass function with $\lambda = \alpha$. The potential in the y-representation reads as

$$V_*(y) = \left[V_{\rm osc} \circ s_R^{-1}\right](y) = \frac{\hbar\omega_0}{2}\sinh^2(\alpha y). \tag{45}$$

Here, the (dimensionless) Schrödinger equation to solve is

$$\left[-\frac{\mathrm{d}^2}{\mathrm{dy}^2} + \sinh^2 \mathrm{y}\right]\varphi = 2\mathrm{E}\varphi. \tag{46}$$

As in the previous case, the energy quantization condition (41) gives an accurate approximation to the eigenvalues E of the energy. With the classical turning points $\pm y_0 = \pm\mathrm{arcsinh}(\sqrt{2\mathrm{E}})/\alpha$, one arrives at the following transcendental equation:

$$\sqrt{2\mathrm{E}}F_E\left(\mathrm{iarcsinh}\sqrt{2\mathrm{E}}\,\middle|-\frac{1}{2\mathrm{E}}\right) = \mathrm{i}\frac{\pi}{2}(k + 1/2), \tag{47}$$

where

$$F_E(\varphi \mid m) = \int_0^{\varphi}(1 - m\sin^2\theta)^{1/2}\,\mathrm{d}\theta$$

is the *elliptic integral of the second kind* (see, e.g. [46]). The roots E_k of (47) can be evaluated numerically by using conventional algorithms. In table 2 we show some of the first values of E_k compared with those obtained from a direct, numerical integration of the Schrödinger equation (46). The corresponding probability densities $|\varphi_*(\mathrm{y})|^2$ are plotted in figure 4, contrasted with their partners $|\psi(x)|^2$ in the x-representation.

In this case the geometry of the curves $V_*(y)$ and $V_{\rm osc}(y)$ is in correspondence with the fact that all the energy eigenvalues of V_* are above the related energy levels of $V_{\rm osc}$. Indeed,

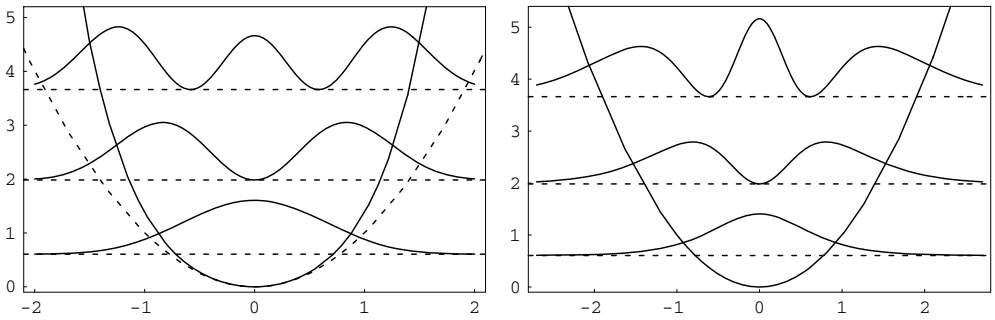


Figure 4. Left: the potential $\sinh^2(y)$ and the probability densities of its three first wavefunctions together with the corresponding energy levels. The oscillator potential curve (dashed) is included as a reference. Right: probability densities of the first three wavefunctions of a second kind, a position-dependent mass oscillator which shares its spectrum with the $\sinh^2(y)$ potential. In both cases, the vertical and horizontal axes are respectively in $\hbar\omega_0$ and dimensionless units.

Table 2. The first ten energy levels of the potential $\sinh^2(y)$ calculated numerically from the WKB transcendental equation (47) and directly from the Schrödinger equation (46).

	E_k in $\hbar w_0$ units	
k	WKB	Schrödinger
0	0.556 44	0.605 71
1	1.944 82	1.983 68
2	3.628 13	3.662 50
3	5.561 79	5.593 65
4	7.719 41	7.749 48
5	10.082 92	10.111 65
6	12.638 90	12.666 57
7	15.376 83	15.403 65
8	18.288 21	18.314 31
9	21.365 92	21.391 41

around the origin one has $V_* \gtrsim V_{\rm osc}$, so that $E_0 \gtrsim 0.5\hbar\omega_0$, as expected (see table 2). For an arbitrary excited level E_k, the distortion is as strong as the speed of the growing up of $V_* - V_{\rm osc}$. In conclusion, the oscillator of the second kind $(V_{\rm osc}, m_R)$ shares its spectrum with a particle of mass m_0 subject to the $\sinh^2$ potential (see figure 4). The spectrum, in turn, is a strongly distorted version of the conventional oscillator's one.

3.2.3. The squeezed oscillator. Let us consider the potential

$$V_{\rm sq}(x) = \frac{\hbar\omega_0}{8}\left\{\left[\frac{1}{x_0+\alpha x} - (x_0+\alpha x)\right]^2 + 2(1-\sqrt{2})\right\}, \qquad x \geqslant t_0, \quad (48)$$

with α defined in (33), x_0 a dimensionless constant and $\mathrm{Dom}(V_{\rm sq}) = [t_0, +\infty)$. This potential is often referred to as a 'singular oscillator' because of its singularity at $x = t_0$. The conventional expression $V_{\rm sq}(x) = m_0\omega_0^2\left(\frac{x^2}{2} + \frac{g^2}{x^2}\right)$, with g being in units of the square of distance and shifted by $-\sqrt{2}\hbar\omega_0/4$, is recovered from (48) with $x_0 + \alpha x = z$ and $\alpha^2 g = \sqrt{2}/4$. Here, we prefer to call it the *sqeezed oscillator* because its domain of definition is the result of a lhs 'squeezing'

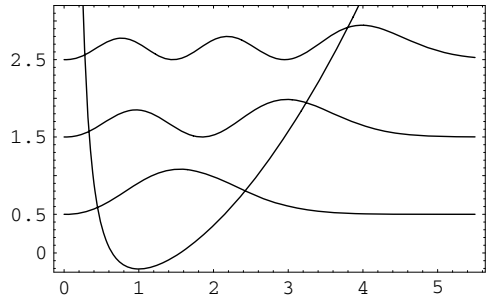


Figure 5. The squeezed oscillator (48) and its first three probability densities. The vertical and horizontal axes are in $\hbar\omega_0$ and dimensionless units respectively.

of $\mathbb{R}$ in terms of s^{-1}, as was established in the previous sections. If the mass function $m(x)$ is a constant m_0, the involved (dimensionless) Schrödinger equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\varphi+\frac{1}{4}\left[\left(\frac{1}{z}-z\right)^2-2(\sqrt{2}-1)\right]\varphi=2\mathrm{E}\varphi,\qquad z=x_0+\alpha x,\tag{49}$$

can be solved in terms of confluent hypergeometric functions by means of the appropriate transformation (see, e.g. [20]). Indeed, the mapping $\varphi\to z^{\ell}\,\mathrm{e}^{-z^2/4}u(z)$, $z\mapsto\sqrt{2\chi}$, leads to the following Kummer equation:

$$\chi\frac{\mathrm{d}^2}{\mathrm{d}\chi^2}u+\left(\frac{2+\sqrt{2}}{2}-\chi\right)\frac{\mathrm{d}}{\mathrm{d}\chi}u-\left(\frac{1}{2}-\mathrm{E}\right)u=0\tag{50}$$

with $\ell=(1+\sqrt{2})/2$. Thereby, the physical solutions for $E_n=\hbar\omega_0(n+1/2)$, $n=0,1,2,\ldots$, read as

$$\begin{aligned}\varphi_n(z)&=C_n z^{\frac{1+\sqrt{2}}{2}}\mathrm{e}^{-\frac{z^2}{4}}{}_1F_1\left(-n,1+\frac{1}{\sqrt{2}},\frac{z^2}{2}\right)\\&=\left(\frac{n!}{2^{1/\sqrt{2}}\Gamma(n+1+1/\sqrt{2})}\right)^{1/2}z^{\frac{1+\sqrt{2}}{2}}\mathrm{e}^{-\frac{z^2}{4}}L_n^{(1/\sqrt{2})}(z^2/2)\end{aligned}\tag{51}$$

with $L_n^{(\alpha)}(x)$ being the *generalized Laguerre polynomials* [46] so that for a constant-mass quantum system, the one-dimensional potential (48) shares its spectrum with the conventional linear harmonic oscillator. Moreover, it is well known that formulae (51) can also be algebraically obtained in terms of the $su(1,1)$ Lie algebra (see, e.g. [30], pp 217). The squeezed oscillator is shown in figure 5, together with some of the corresponding probability densities.

If $m(x)$ is not a constant then the energy spectrum of the quantum system is modified, as we have previously verified. According to table 1, any of the masses (21), (26) or (30) allows the mapping to the y-space. First let us consider the case $m=m_{(0)}$. The potential (48) is mapped to the following one:

$$V_*(y_{(0)})=\left[V_{\mathrm{sq}}\circ s_{(0)}^{-1}\right](y_{(0)})=\frac{\hbar\omega_0}{2}\left[\sinh^2(\alpha y_{(0)})+\frac{1-\sqrt{2}}{2}\right]\tag{52}$$

which, up to an additive constant, is the same as the potential reported in equation (45). Thereby, we have shown that a quantum system endowed with mass $m_{(0)}(x)$ and acted on by the oscillator-like potential (48) shares its spectrum with a particle of mass m_0 which is under the action of the potential $\sinh^2(\mathrm{y}_{(0)})$. In comparison with a constant mass quantum oscillator,

we realize that the presence of $m_{(0)}(x)$ distorts the ground-energy level of the oscillator-like system from 0.5 to ≈0.6 energy units ($\hbar\omega_0$), the second one from 1.5 to ≈1.9, and so on. The higher the level of excitation of the system the stronger the distortion of the spectrum.

Note that the system studied in section 3.2.2 behaves in a similar manner, so there exists a clear relationship between position-dependent mass systems: different masses combined with appropriate interactions give rise to the same spectrum. In this case, the oscillators of the second kind defined by the pairs $(V_{\rm osc}, m_R)$ and $(V_{\rm sq}, m_{(0)})$ are isospectral (see table 2). On the other hand, we have another pair of oscillators of the second kind $(V_{\rm sq}, m_{(n)})$ and $(V_{\rm sq}, m_R)$, which are respectively isospectral with the constant-mass potentials

$$V_*(y_{(n)}) = \frac{\hbar\omega_0}{8}\left\{\left[\left(\frac{2n+1}{\alpha y_{(n)}}\right)^{2n+1} - \left(\frac{\alpha y_{(n)}}{2n+1}\right)^{2n+1}\right]^2 + 2(1-\sqrt{2})\right\}, \qquad n \in \mathbb{N}, \tag{53}$$

and

$$V_*(y) = \frac{\hbar\omega_0}{8}\left\{\left[\frac{1}{x_0+\sinh\alpha y} - (x_0+\sinh\alpha y)\right]^2 + 2(1-\sqrt{2})\right\}. \tag{54}$$

Each one of these last potentials shows a spectrum which is a distorted version of $\mathrm{Sp}(V_{\rm sq}) = \mathrm{Sp}(V_{\rm osc})$. In summary, given an interaction represented by $V(x)$, the spectrum of a position-dependent mass quantum system is a distorted version of the spectrum of a particle of mass m_0 subject to the same interaction. As we have realized, the degree of distortion depends directly on the explicit position dependence of the involved mass.

4. Factorization and coherent states

Once we have constructed the solvable position-dependent mass Hamiltonians H_a, one can look for the appropriate factorization operators. The presence of $m(x)$ in H_a makes necessary a refinement of the factorization (see, e.g. [20, 47]). As usual, the factorization operators intertwine the initial Hamiltonian with a set of new exactly solvable energy-like operators $\widetilde{H}_a$ [15, 16]. However, in general they do not act as ladder operators on the eigenfunctions of either H_a or $\widetilde{H}_a$. In the case of position-dependent mass oscillators of the first kind, the factorization operators act in a ladder form if their commutator is the appropriate constant. Then, as we are going to show, one is able to construct a set of position-dependent mass coherent states.

4.1. The position-dependent mass factorization

Let A and B be the following operators:

$$A = -\frac{\mathrm{i}}{\sqrt{2}}m^a P m^b + \beta, \qquad B = \frac{\mathrm{i}}{\sqrt{2}}m^b P m^a + \beta, \qquad A^\dagger = B, \tag{55}$$

with β being a function of the position operator X. We want to work with A and B as the factorization operators of H_a. In this regard, it is important to stress that most of the literature pays attention to a specific ordering of m and P. Namely, it is usual to take $a = 0$ and $b = -1/2$ so that the kinetic part of H_a reads as $\frac{1}{2}P\frac{1}{m}P$, with the corresponding simplification of A and B (see, e.g. [21, 22, 28]). Here, we shall use the operators (55) with no *a priori* assumption on the ordering of m and P. In this way, the results already reported will be included as particular cases.

If A and B factorize the Hamiltonian (1) in a *refined* way [47], then one has

$$H_a = AB + \epsilon, \tag{56}$$

and β fulfills a Riccati equation in the position representation:

$$V-\epsilon=\frac{\hbar}{\sqrt{2m}}\left[2\left(a+\frac{1}{4}\right)\left(\frac{m'}{m}\right)\beta-\beta'\right]+\beta^2, \tag{57}$$

where ϵ is a constant (in energy units) to be fixed. For arbitrary m and β the product between the factorization operators obeys the commutation rule:

$$[A,B]=-\frac{\hbar^2}{m^3}\left(a+\frac{1}{4}\right)\left[mm''-\frac{3(m')^2}{2}\right]-\frac{2\hbar}{\sqrt{2m}}\beta'. \tag{58}$$

Therefore, we have a new operator $\widetilde{H}_a$, defined as follows:

$$\widetilde{H}_a\equiv BA+\epsilon=K_a+\widetilde{V},\qquad \widetilde{V}:=V-[A,B], \tag{59}$$

which is intertwined with H_a by means of the factorization operators:

$$\widetilde{H}_aB=BH_a,\qquad H_aA=A\widetilde{H}_a. \tag{60}$$

The relevance of these last relationships is clear by noting that, if ψ is an eigenfunction of H_a with eigenvalue E (see equation (5)), then $\widetilde{\psi}\propto B\psi\neq 0$ solves the new eigenvalue equation

$$\widetilde{H}_a\widetilde{\psi}(x)=E\widetilde{\psi}(x). \tag{61}$$

Moreover, it is easy to verify that a normalized wavefunction ψ leads to $|\widetilde{\psi}|^2\propto E-\epsilon$. Then, the new set $\{\widetilde{\psi}=B\psi/(E-\epsilon)^{1/2}|E\neq\epsilon\}$ consists of normalized eigenfunctions of $\widetilde{H}_a$ belonging to the eigenvalues $\{E\}=\mathrm{Sp}(H_a)$. Now, let $\widetilde{\psi}_\epsilon$ be a function which is orthogonal to the set $\{\widetilde{\psi}\}$, i.e., $(\widetilde{\psi},\widetilde{\psi}_\epsilon)\propto(\psi,A\widetilde{\psi}_\epsilon)=0$. Since $\psi\neq 0$ we have $A\widetilde{\psi}_\epsilon=0$ and necessarily $\widetilde{H}_a\widetilde{\psi}_\epsilon=\epsilon\widetilde{\psi}_\epsilon$. The involved solution reads as

$$\widetilde{\psi}_\epsilon=C_\epsilon m^{a+1/2}\exp\left[\frac{\sqrt{2}}{\hbar}\int^x m^{1/2}\beta\,\mathrm{d}r\right] \tag{62}$$

with C_ϵ being a constant of integration. If $(\widetilde{\psi}_\epsilon,\widetilde{\psi}_\epsilon)<\infty$, then $\mathrm{Sp}(\widetilde{H}_a)=\mathrm{Sp}(H_a)\cup\{\epsilon\}$.

The previous derivations considered $\epsilon\notin\mathrm{Sp}(H_a)$. To include the case $\epsilon\in\mathrm{Sp}(H_a)$ let us assume that the solution of $B\psi_M=0$, given by

$$\psi_M=C_MC_\epsilon m^{1/2}(\widetilde{\psi}_\epsilon)^{-1}, \tag{63}$$

is a square-integrable function. In this way ψ_M is the wavefunction of H_a belonging to the eigenvalue $E=\epsilon$. As a consequence, there is no element in $\{\widetilde{\psi}\}$ constructed from ψ_M via relationships (60). The corresponding function $\widetilde{\psi}_M$ must be obtained as a solution of $BA\widetilde{\psi}_M=0$ (see, equations (59) and (61)). There are two possible cases, as follows:

(1) If $A\widetilde{\psi}_M=0$, then $\widetilde{\psi}_M$ has the same form as the function defined in (62). However, if $\psi_M\in L^2(\mathcal{D}^{(a)})$, from equation (63) one notes that $\widetilde{\psi}_M\propto m^{1/2}/\psi_M$ is not square integrable.

(2) If $A\widetilde{\psi}_M\neq 0$ and $B(A\widetilde{\psi}_M)=0$, one can take $A\widetilde{\psi}_M=\psi_M$ such that $B(\psi_M)=0$. Then, because ψ and ψ_M are orthogonal, we have $(\widetilde{\psi},\widetilde{\psi}_M)=(\psi,A\widetilde{\psi}_M)=(\psi,\psi_M)=0$. Thus, $\widetilde{\psi}$ and $\widetilde{\psi}_M$ are orthogonal and $\mathrm{Sp}(\widetilde{H}_a)=\mathrm{Sp}(H_a)$, with $\epsilon\in\mathrm{Sp}(H_a)$.

At this stage, it is important to stress that $\mathrm{Sp}(H_a)$ and $\{\psi\}$ can be obtained by means of the transformations introduced in section 2. Thereby, we get a wide family of isospectral operators if, for instance, $H^{(a_n)}$ is the MDNT-Hamiltonian defined in (28). That is, we have

$$\mathrm{Sp}\big(\widetilde{H}_{a_n}\big)=\mathrm{Sp}\big(H_{a_n}\big)\rightleftarrows\mathrm{Sp}(H^{(a_n)})=\mathrm{Sp}(\widetilde{H}^{(a_n)}), \tag{64}$$

where $\widetilde{H}^{(a_n)}$ is the Hamiltonian intertwined with $H^{(a_n)}$ in the $y_{(n)}$-representation. The same can be said about the MINT-Hamiltonian H defined in (29). In this context, it will be profitable to decompose the commutator (58) in the MDNT and MINT cases:

$$[A, B] = \begin{cases} -\left[\dfrac{\hbar}{m^{3/2}}\left(a+\dfrac{1}{4}\right)m'\right]^2 - \sqrt{\dfrac{2\hbar^2}{m}}\beta' & \text{(MDNT)} \\ -\sqrt{\dfrac{2\hbar^2}{m}}\beta' & \text{(MINT).} \end{cases} \tag{65}$$

4.2. Position-dependent mass ladder operators

Let us consider the Hamiltonian of an oscillator of the first kind H_a. In advance we know that $\mathrm{Sp}(H_a) = \{E_n = \hbar\omega_0(n+1/2)\}_{n=0}^{+\infty}$, whether we deal with the MDNT or the MINT case (see section 3). To get the simplest form for the corresponding annihilation and creation operators, let us take $[A, B] = -\hbar\omega_0$. Then we have $\widetilde{H}_a = H_a + \hbar\omega_0$. That is, $\widetilde{H}_a$ differs from H_a only in the zero of the potential. This physical equivalence and the intertwining relationships (60) make clear the roles played by the factorization operators:

$$A(H_a + \hbar\omega_0) = H_a A, \qquad B(H_a - \hbar\omega_0) = H_a B. \tag{66}$$

Therefore, $A\psi_n \propto \psi_{n+1}$ and $B\psi_n \propto \psi_{n-1}$ if $H_a\psi_n = E_n\psi_n$. Now, the substitution of $[A, B] = -\hbar\omega_0$ into (58) leads to the following β-function:

$$\beta = \frac{\omega_0}{\sqrt{2}}\int^x m^{1/2}\,\mathrm{d}r - \frac{\hbar}{\sqrt{2}}\left(a+\frac{1}{4}\right)\left(\frac{m'}{m^{3/2}}\right) + \beta_0. \tag{67}$$

Here, β_0 is an integration constant which will be omitted in the following. The identification $\epsilon = \hbar\omega_0/2$, after introducing (67) in the Riccati equation (57), allows us to write the potential V in terms of the β-function:

$$V = \beta^2 + \frac{2\hbar}{\sqrt{2m^3}}\left(a+\frac{1}{4}\right)\left\{m'\beta + \frac{\hbar}{2\sqrt{2m^3}}\left[mm'' - \frac{3}{2}(m')^2\right]\right\}. \tag{68}$$

The straightforward calculation shows that this last expression is reduced in both the MDNT and the MINT cases to the same simple form

$$V = \frac{\omega_0^2}{2}\left[\int^x m^{1/2}\,\mathrm{d}r\right]^2. \tag{69}$$

Note that this expression for the potential is consistent with our transformations in sections 2 and 3. Indeed, since $V_*(y) = \frac{m_0\omega_0^2}{2}y^2$ has been given as the initial potential, its x-representation reads as

$$V(x) = [V_* \circ s](x) = V_*(s(x)) = \frac{m_0\omega_0^2}{2}(s(x))^2 \tag{70}$$

with $y = s(x)$ given in (13). In summary, we have shown that A and B, as they are defined in (55)–(57), are nothing but creation and annihilation operators if their commutator (58) is constrained to be a constant equal to the separation between the energy levels of H_a. The same condition allows us to identify a quadratic potential V, expressed in terms of the mass function, which is consistent with the transformations defined in the previous sections.

4.3. Position-dependent mass coherent states

To take full advantage of the results derived in the previous sections, let us rewrite the factorization operators as follows (compare with [21, 22, 28]):

$$A = -\frac{\hbar}{\sqrt{2m}}\left[\frac{\mathrm{d}}{\mathrm{d}x} - \frac{(\ln m)'}{4}\right] + \frac{\omega_0}{\sqrt{2}}\int^x m^{1/2}\,\mathrm{d}r, \tag{71}$$

$$B = \frac{\hbar}{\sqrt{2m}}\left[\frac{\mathrm{d}}{\mathrm{d}x} - \frac{(\ln m)'}{4}\right] + \frac{\omega_0}{\sqrt{2}}\int^x m^{1/2}\,\mathrm{d}r, \tag{72}$$

where we have used (55) and (67). The operator B in the y-space is then given by

$$B_* = \frac{\hbar}{\sqrt{2m_0}}\frac{\mathrm{d}}{\mathrm{d}y} + \left(\frac{\omega_0^2 m_0}{2}\right)^{1/2} y - \frac{\hbar}{\sqrt{32m_0}}\left(\frac{d\ln m_*}{\mathrm{d}y}\right) \tag{73}$$

and a similar expression for A_*, obtained from (73) by changing the sign of the first and the third terms. Finally, in the dimensionless notation of equation (33) we get

$$\mathrm{B}_* = \frac{\mathrm{d}}{\mathrm{d}y} + \mathrm{y} - \left(\frac{\mathrm{d}}{\mathrm{d}y}\ln m_*^{1/4}\right), \qquad B_* = \left(\frac{\hbar\omega_0}{2}\right)^{1/2}\mathrm{B}_* \tag{74}$$

and $A_* = \mathrm{A}_*\sqrt{\hbar\omega_0/2}$. Then, from (55) we have

$$\mathrm{H}_{a*} = (2/\hbar\omega_0)H_{a*} = \mathrm{A}_*\mathrm{B}_* + 1, \qquad [\mathrm{A}_*, \mathrm{B}_*] = -2. \tag{75}$$

The action of H_{a*}, A_* and B_* on ψ in the dimensionless y-representation is as follows:

$$\mathrm{H}_{a*}\psi_* = J_*^{1/2}(-\ddot{\varphi} + y^2\varphi), \qquad \mathrm{A}_*\psi_* = J_*^{1/2}a_+\varphi_*, \qquad \mathrm{B}_*\psi_* = J_*^{1/2}a_-\varphi_*. \tag{76}$$

The Jacobian J is defined in equation (16) and a_- (a_+) is the conventional annihilation (creation) operator of the linear oscillator in the y-representation.

$$a_- := \frac{\mathrm{d}}{\mathrm{d}y} + \mathrm{y}, \quad (a_+)^\dagger = a_-, \qquad [a_-, a_+] = 2, \qquad a_+a_- = 2N \tag{77}$$

with N being the Fock's number operator. Hereafter, we shall omit the '*-notation'.

In order to construct a set of coherent states as eigenfunctions of B, we first take an arbitrary linear combination Θ of the wavefunctions ψ_n associated with H_a:

$$\Theta = \sum_{k=0}^{\infty} c_k\psi_k. \tag{78}$$

The action of B on this last function reads as

$$\mathrm{B}\Theta = J^{1/2}\sum_{k=0}^{\infty} c_k\sqrt{2k}\varphi_{k-1}. \tag{79}$$

We look for the functions Θ fulfilling $\mathrm{B}\Theta = z\Theta$, $z \in \mathbb{C}$. The straightforward calculation leads to a recurrence relation which is satisfied by the coefficients c_k. The root is found to be $c_k = z^k c_0/\sqrt{k!2^k}$. As usual, the coefficient c_0 is fixed by the normalization of Θ and we finally arrive at the familiar expression:

$$\Theta_z = \mathrm{e}^{-\frac{|z|^2}{4}}\sum_{k=0}^{\infty}\frac{z^k}{\sqrt{2^k k!}}\psi_k = J^{\frac{1}{2}}\,\mathrm{e}^{-\frac{|z|^2}{4}}\,\mathrm{e}^{\frac{za_+}{2}}\varphi_0 \equiv J^{\frac{1}{2}}\,\mathrm{e}^{-\frac{|z|^2}{4}}\,\mathrm{e}^{\frac{za_+}{2}}\,\mathrm{e}^{-\frac{\bar{z}a_-}{2}}\varphi_0 = J^{\frac{1}{2}}D(z)\varphi_0, \tag{80}$$

where $\bar{z}$ stands for the complex conjugation of z and we have used the Baker–Campbell–Hausdorff formula $e^A e^B = \exp\left(A + B + \frac{1}{2}[A, B]\right)$, with $[A, [A, B]] = [B, [A, B]] = 0$, to recover the displacement operator

$$D(z) := \mathrm{e}^{\frac{za_+ - \bar{z}a_-}{2}} = \mathrm{e}^{za_+/2}\,\mathrm{e}^{-\bar{z}a_-/2}\,\mathrm{e}^{-|z|^2/4}. \tag{81}$$

Thereby, since $\theta_z(y) := D(z)\varphi_0(y)$ is a conventional constant mass coherent state in the y-space we conclude that its partner in the x-coordinates $\Theta_z = J^{1/2}\theta_z$ is a position-dependent mass coherent state, defined in terms of the annihilation operator B. Explicitly, $\Theta_z(x)$ is given by

$$\Theta_z(x) = \left(\frac{m(x)\,\mathrm{e}^{-|z|^2}}{m_0}\right)^{1/4} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{2^k k!}} \varphi_k(s(x)). \tag{82}$$

This last result is the general expression for the coherent states of any of the oscillators of the first kind introduced in section 3.1. In this context, let one of these oscillators be in the state Θ_z. The probability of getting $\mathrm{E}_n = 2n + 1$ as the result of a measurement of the energy is ruled by the Poisson distribution:

$$\mathcal{P}_n(\Theta_z) \equiv |(\psi_n, \Theta_z)|^2 = \frac{|z|^{2n}}{2^n n!}\,\mathrm{e}^{-|z|^2/2}. \tag{83}$$

The mean value $\langle \mathrm{H}_a \rangle_z$ is then given by

$$\langle \mathrm{H}_a \rangle_z \equiv (\Theta_z, \mathrm{H}_a\Theta_z) = \sum_{k=0}^{\infty} \mathcal{P}_k(\Theta_z)\mathrm{E}_k = |z|^2 + 1, \tag{84}$$

where we have used equation (75). In the same manner we find $\left\langle \mathrm{H}_a^2 \right\rangle_z = |z|^4 + 4|z|^2 + 1$, so that $\Delta \mathrm{H}_a = |z|\sqrt{2}$. Hence, for very large $|z|$ one gets $\Delta \mathrm{H}_a \ll \langle \mathrm{H}_a \rangle_z$ and the relative value of the energy of the state Θ_z is well defined, as usual for the Glauber states. It is also simple to verify that the product of the root-mean-square deviations ΔP and ΔY is minimized. Notice that $Y = s(X)$, with s being the transformation defined in (9). Thereby, the states Θ_z of equation (82) minimize the uncertainty relation between position and momentum in a relaxed form. In conclusion, the z-parameterized functions (82) are the coherent states belonging to a wide class of position-dependent mass oscillators of the first kind.

5. Concluding remarks

We have studied the problem of solving the Schrödinger equation for an arbitrary position-dependent mass system. Our approach is useful to face two general physical situations. In the first one we look for the interaction which must be applied to a mass $m(x)$ to supply it with a particular spectrum of energies. The second physical situation corresponds to the case in which one is interested in a given position-dependent mass $m(x)$, subject to a particular interaction $V(x)$ rather than in the recovery of a specific spectrum. For arbitrary orderings of $m(x)$ and P in the Hamiltonian, diverse general expressions for $m(x)$ were derived as a consequence of mapping the original Schrödinger equation to a conventional constant mass one. It was also found that the transformation is rather simple for the very special ordering $m^{-1/4}Pm^{-1/2}Pm^{-1/4}$ in the kinetic part of the Hamiltonian. In contradistinction with [21, 22, 28], we showed that a position-dependent mass Hamiltonian can be factorized as the product of two mutually adjoint operators with no *a priori* assumptions on the ordering of m and P.

In particular, two kinds of position-dependent mass oscillators were analyzed. The first one is defined to be isospectral with the quantum oscillator of mass m_0, no matter what the explicit form of $m(x)$ and $V(x)$ is. The oscillators of the second kind exhibit spectra different from the equidistant energies $\hbar\omega_0(n + 1/2)$ and correspond to a particular mass $m(x)$ subjected to the harmonic oscillator potential. Results include the singular oscillator as well as confining odd-root-law, $\ln^2$ and $\sinh^2$-like interactions. The special case of a particle of

mass m_0 in a confining even-power-law potential was found to be isospectral to a system of mass $m(x) \propto x^{-4n/(2n+1)}$, subject to the action of the harmonic oscillator potential. The factorization operators were then selected to work as ladders in the space of the position-dependent mass oscillators. Finally, the coherent states corresponding to oscillators of the first kind were explicitly constructed as eigenvectors of the annihilation operator. These new CS have the same analytical form as the Glauber states and minimize a relaxed version of the position–momentum uncertainty principle.

Of special interest, the singular oscillator $V_{\rm sq}$, defined in equation (48) and referred to in section 3.2.3 as the *squeezed oscillator*, exhibits CS connected with the $su(1, 1)$ Lie algebra if the mass is a constant [30]. Particular cases of the mass function have been shown to preserve the $su(1, 1)$ spectrum structure of $V_{\rm sq}$ [28]. The same is true for any of the masses derived in this paper. Thereby, it is sound to construct position-dependent mass $su(1, 1)$-like CS. Following [30], such a result could be applied to get a better understanding of the physics of N interacting particles (work in this direction will be published elsewhere [48]). Other physically interesting systems can be analyzed in the corresponding manner once the dynamical algebra is given. Special attention must be drawn to the Susy nonlinear algebras engaged with infinite point spectra. If the energy levels can be obtained by a function of its index $E_n = E(n)$, then one can distinguish between *natural* and *linear* algebras of the Susy system. To each one of these algebras, there exists a companion set of CS [32]. Then, besides the systems discussed above, it would be also interesting to analyze the position-dependent mass CS belonging to higher order Susy partners of potentials like the Pöschl–Teller one [49, 50] (see also [36]). Results on this matter are in progress.

Acknowledgments

The support of CONACyT project 24333-50766-F and IPN grants COFAA and EDI is acknowledged. ORO is grateful to Y Gurevich for enlightening comments. SCyC thanks the members of Physics Department, Cinvestav for kind hospitality.

Note added in proof. After this paper was accepted for publication we were advised of the work by G X Ju, C Y Cai and Z Z Ren in arXiv:0707.3259 (hereafter 'Ju paper'). There, the authors discuss the factorization of the Hamiltonian H_a for a specific value of a. Namely, they necessarily arrive at the constraint $a = -1/4$ ($\beta = -1/2$ in the notation of the Ju paper). In this sense, the approach of Ju *et al* adheres to the one used in e.g. [21, 22, 28] and their results are recoverable from our approach concerning the oscillators of the first kind. Nevertheless, the Ju paper reports on the study of CS in connection with position-dependent mass oscillators. Ju and co-workers calculate the root-mean-square (rms) deviations for a pair of functions $\mu(x)$ and $\hat{\pi}(x)$—see equations (26)–(28) of the Ju paper—the product of which can be put in correspondence with our relaxed version of the uncertainty principle if the identifications $y \propto \mu(x)$ and $P_* \propto \hat{\pi}(x)$ are feasible.

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